Renormalization Method and its Economic Applications

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Abstract:

The purpose of this paper is to give new insights of the method of Helleman (1980) in the context of macrodynamics. This method explains how a difference equation can be locally studied from the Feigenbaum equation in the case of a constant Jacobian matrix. First we introduce this technique. Second we apply it in two models: the model of Matsuyama (1999) and the model of Kaldor (1957). Finally we present an extension of the technique in the case of non constant (linear) Jacobian matrix and apply this extension in the model of Médio (1992).

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1- Introduction

Non linear dynamics techniques are nowadays the commonly used techniques in macrodynamics. It is traditional that the evolution of an economy is studied from the qualitative analysis around the stationary equilibrium of this economy. If it is quite easy to proceed to the analysis of the local stability of this equilibrium, it is more difficult to have results dealing with the global stability of this equilibrium. Nevertheless, the local analysis can bring interesting results around the stationary equilibrium. For example if an equilibrium can lose its stability and becomes unstable, then the emergence of a two-period cycle can be possible. This particular phenomenon is called “bifurcation”. The application of these techniques has given new insights to macrodynamics through the analysis of endogenous fluctuations.

The emergence of endogenous fluctuations in pure-exchange overlapping generations (OLG) models has been well established since the papers of Benhabib and Day (1982) and of Grandmont (1985). The crucial assumption in their seminal papers is the presence of nonlinearities in agents’ behaviour. Unfortunately, their economic interpretation is hard to justify, as it requires a strong income effect. Economists have then tried to go over that problem by expanding the original model in two ways: either by introducing economic policies, or by developing the sector of production. In the latter case, the model is more sophisticated. In the original work, the difference equation of the model is a ‘simple’ logistic equation. When one introduces a specific sector of production, the difference equation is second-order and one needs to show a Hopf bifurcation in the model to find endogenous fluctuations. Reichlin (1986) was the first to show the occurrence of the latter in an OLG model with production in the case of an elastic labour supply and Leontief technology. This kind of economic model was later adopted by Grandmont (1993), Médio (1992), Médio and Negroni (1996), Reichlin (1992). In particular, Médio (1992) and Hommes *et al.* (1994) investigate the model through numerical simulations in order to detect complex dynamics or to determine global dynamic behaviour. In these papers, the period-doubling bifurcations are regarded as a typical route to randomness. Indeed it is not easy to rigorously verify the occurrence of period-doubling paths in a given economic model. De Vilder (1996) applies homoclinic bifurcation theory in Reichlin’s model. He can then show the possibility of cascades of infinitely many period-doubling bifurcations, leading to strange attractors. We propose here a complementary and attractive technique because of its tractability. The idea of this technique arises out of the reading of Médio’s (1992) book. The cascades of bifurcations Médio finds remind us of those of the logistic function. We then wish to apply a rigorous technique, the renormalization procedure, to show their emergence.

This technique is well-known in physics and was created by Helleman (1980). It allows us to compare the dynamics of any difference equation with those of the Feigenbaum equation under some assumptions. Helleman’s approach has lots of advantages. Among them, the critical values characterising the transitions of dynamics to complexity are easily interpreted in economic terms.

Our paper provides interesting conclusions in methodological terms and in an economic point of view. In methodological terms, our results are twofold. On the one hand, we reveal a technique to obtain locally a relationship between a map and the Feigenbaum equation, without having recourse to Helleman's constant Jacobian assumption (cf. Lichtenberg and Lieberman (1992)). On the other hand, we develop the idea of "critical functions of bifurcations" implicitly created by Helleman via the Feigenbaum equation.
From an economic point of view, our conclusions are twofold. First we are able to apply easily the method of Helleman in Matsuyama's model (1999) and Kaldor's model (1957). Second we strengthen the results of Reichlin (1992) when we apply our generalised method in his model. Indeed the productivity of factors, central part in our analysis, allows us to show the possibility of endogenous fluctuations.

The paper is organised as follows. Section 2 deals with the Feigenbaum equation and its renormalisation procedure. Section 3 presents the topic of critical functions of bifurcations related to the Feigenbaum equation and we apply the technique in the model of Matsuyama (1999) and the model of Kaldor (1957). Section 4 generalises the technique in the case of a non constant Jacobian (and linear) at the stationary equilibrium. We then apply this extension in the model of Médio (1992) in Section 5. We are then able to expose the economic conclusions from our own numerical examples.

2- Renormalization Procedure

We focus our attention on the Feigenbaum equation. In the first subsection we present the renormalization procedure of the Feigenbaum equation. In the subsection thereafter we illustrate this procedure via two examples: the logistic case and the Hamiltonian case. Finally we summarise Helleman's method.

2.1- The renormalization of the Feigenbaum equation

Definition 1 The second-order difference equation denoted as:

$$\Delta_{t+1} + B\Delta_{t-1} = 2\mu\Delta_t + 2\Delta_t^2 \text{ for all } t \in N \tag{1}$$

where μ and B are two real parameters, is known as the Feigenbaum equation.

Rewrite (1) by letting $\varepsilon_t = \Delta_{t-1}$. We then obtain the system (S1):

$$\begin{pmatrix} \Delta_{t+1} \\ \varepsilon_{t+1} \end{pmatrix} = \begin{pmatrix} 2\mu & -B \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_t \\ \varepsilon_t \end{pmatrix} + \begin{pmatrix} 2\Delta_t^2 \\ 0 \end{pmatrix} \tag{S1}$$

The point $(0,0)$ is a stationary point of (S1). The theorem of Hartman (1964) allows us to show that this equilibrium loses its stability when $\mu < -(1+B)/2$. By mirror-symmetry, we can find another critical value of bifurcation, $\mu > 3(1+B)/2$ (see Appendix A). From these two values appear two symmetric Flip-cascades of bifurcations.

Recall that we consider the case where the system (S1) is dissipative, i.e. $|B| < 1$. Let $(\xi_t)_{t\in N}$ the path of a two-period cycle of (1) such that:

$$\begin{cases} \xi_{2t+1} = \xi_1 \\ \xi_{2t} = \xi_0 \end{cases} \text{ for all } t \in N$$

Denote by $\delta_t = \Delta_t - \xi_t$, the distance between the path considered and the two-period cycle. By substituting δ_t into (1), we obtain:

$$\delta_{t+1} + B\delta_{t-1} = (2\mu + 4\xi_t)\delta_t + 2\,\delta_t^2 \text{ for all } t \in N \qquad (2)$$

By using the renormalization procedure (the complete computation is available in Appendix A), we are then able to obtain the Feigenbaum equation from (2):

$$\tilde{\delta}_{k+1} + B'\tilde{\delta}_{k-1} = 2\mu'\tilde{\delta}_k + 2\,\tilde{\delta}_k^2 \text{ for all } k \in N \qquad (3)$$

where $\mu'(B') = -2\mu^2(B) + 2(1+B)\mu(B) + 2B^2 + 3B + 2$ and $B' = B^2$.

Every 2^k-period cycle of (3) is a 2^{k+1}-period cycle of equation (1). We can then prove the existence of two recurrent ascendant paths of μ:

$$\mu_k(B^2) = -2\mu_{k+1}{}^2(B) + 2(1+B)\mu_{k+1}(B) + 2B^2 + 3B + 2 \text{ for all } k \in N \qquad (4a)$$

$$\mu_k(B^2) = +2\mu_{k+1}{}^2(B) - 2(1+B)\mu_{k+1}(B) - B^2 - 3B - 1 \text{ for all } k \in N \qquad (4b)$$

Each term of these paths corresponds to the loss of a stable 2^k-period cycle and the occurrence of a 2^{k+1}-period cycle. We can then represent the ascendant paths of μ:

[Insert Figure 1]

The two paths $(\mu_k(B))_{k\in N}$ and $(\overline{\mu}_k(B))_{k\in N}$ are called paths of critical values coming from the renormalization process associated with the Feigenbaum equation.

2.2- Illustration

We can illustrate our previous findings by taking the examples of the logistic and the Hamiltonian.

2.2.1- *The logistic case ($B=0$)*

When $B=0$, we have the logistic equation. The system is then locally dissipative. Its accumulation point verifies $2\mu^2-\mu-2=0$ and is equal to $\mu_\infty=\left(1-\sqrt{17}\right)/4$ where $\mu_0=-1/2$. Another accumulation point can be obtained from the mirror-symmetry, $\bar{\mu}_\infty=1-\mu_\infty=\left(3+\sqrt{17}\right)/4$ where $\bar{\mu}_0=3/2$. Two paths of critical values of bifurcation have been found $\left(\mu_k(B)\right)_{k\in N}$ and $\left(\bar{\mu}_k(B)\right)_{k\in N}$ with $\mu_0=-1/2$ and $\bar{\mu}_0=3/2$.

2.2.2-*The Hamiltonian case ($B=1$)*

When $B=1$, we have the Hamiltonian equation. The system is then locally conservative. Its accumulation point verifies $2\mu^2-3\mu-7=0$ and is equal to $\mu_\infty=\left(3-\sqrt{65}\right)/4$ where $\mu_0=-1$. Another accumulation point can be obtained from the mirror-symmetry, $\bar{\mu}_\infty=1+B-\mu_\infty=\left(5+\sqrt{65}\right)/4$ where $\bar{\mu}_0=3$. Two paths of critical values of bifurcation have been found $\left(\mu_k(B)\right)_{k\in N}$ and $\left(\bar{\mu}_k(B)\right)_{k\in N}$ with $\mu_0=-1$ and $\bar{\mu}_0=3$.

2.3- Abstract of Helleman's approach (1980)

Helleman (1980) shows that a map can be locally equivalent to the Feigenbaum equation thanks to appropriate variable changes under the assumption of a constant Jacobian at all points of the two-dimensional space. This approach considers a dynamic system Φ (continuously differentiable) which describes a forward dynamics:

$$R^2 \to R^2,\ X_t \to \Phi(X_t) = X_{t+1} \text{ for all } t \in N$$

Denote by X^*, the stationary point of Φ. If the eigenvalues of the Jacobian matrix associated with Φ are real and different, then the Feigenbaum equation can be deduced from Φ thanks to appropriate variable transformations:

$$\Delta_{t+1} + B\Delta_{t-1} = 2\mu\Delta_t + 2\Delta_t^2 \text{ for all } t \in N$$

where $\mathrm{Jac}[\Phi(X^*)]$ is the Jacobian matrix of the system at the stationary point, $\mu = \mathrm{Tr}[\mathrm{Jac}(\Phi(\mathrm{X}^*))]/2$ and $B = \mathrm{Det}[\mathrm{Jac}(\Phi(\mathrm{X}^*))]$. We assume that the system is dissipative, i.e. $|B| < 1$.

Denote as $\tilde{X}$, the stationary point of Φ^{2^k}. We can then obtain $\mathrm{Det}\left[\mathrm{Jac}\left(\Phi^{2^k}(\tilde{\mathrm{X}})\right)\right] = B^{2^k}$ and the difference equation for all $k \in N$:

$$\mu_k(B_{k+1}) = -2\mu_{k+1}^2(B_k) + 2(1+B_k)\mu_{k+1}(B_k) + 2B_k^2 + 3B_k + 2 \quad (5)$$

where $B_{k+1} = B_k^2$.

Since $|B| < 1$, then $\lim_{k\to\infty} B_k = 0$ and the accumulation points of the path are given by the expressions (4a) and (4b). These can be found by applying the method of mirror-symmetry. In general, the Jacobian matrix varies a little around a periodic orbit. That is why, when the eigenvalues of the Jacobian matrix are real and different, we can consider that the Feigenbaum equation approximates locally the dynamics of the original map.

3- Critical Functions, Intervals of Qualitative Forecasts and Their Economic Sense

We focus our attention on the critical functions. In the first subsection we present the concept of such functions. In the subsections thereafter by applying this concept in well-known economic models we emphasise the advantages of the renormalization procedure.

3.1- The idea of Critical Functions and the Intervals of Qualitative Forecasts

The renormalization procedure leads us to explain the relationship between successive critical values of bifurcation. Each critical value characterises the loss of a stable 2^k-period cycle and the occurrence of a 2^{k+1}-period cycle when the Jacobian matrix is constant. In other words, the Helleman technique gives the variable $\mu_k\left(B^2\right)$ in function of $\mu_{k+1}(B)$ for a given value of *B*. It is then possible to exhibit $(\mu_k(B))_{k\in N}$ and $(\overline{\mu}_k(B))_{k\in N}$, the two paths associated with the cascades of bifurcation of (1) for a given value of *B*. When we consider a dissipative system with real and different eigenvalues, we are able to obtain a 2^k-period cycle around the stationary point by taking into account rough estimates. However the interpretation of the initial dynamics can be falsified because the calculated values of bifurcation are not necessarily equal to exact values.

In order to refine the qualitative analysis, the determination of critical values of the Jacobian matrix is now important. Consider first:

$$\varphi_k\left(B^2\right) = -2\varphi_{k+1}{}^2(B) + 2(1+B)\varphi_{k+1}(B) + 2B^2 + 3B + 2 \text{ for all } k \in N$$

$$\overline{\varphi}_k\left(B^2\right) = +2\overline{\varphi}_{k+1}{}^2(B) - 2(1+B)\overline{\varphi}_{k+1}(B) - B^2 - 3B - 1 \text{ for all } k \in N$$

Since these two paths of critical values are respectively decreasing and increasing, the solving of second-order equations with which they are associated allows us to establish:

$$\varphi_{k+1}(B) = \frac{1}{2}\left((1+B) - \sqrt{5B^2 + 8B + 5 - 2\varphi_k\left(B^2\right)}\right)$$

and $$\overline{\varphi}_{k+1}(B) = \frac{1}{2}\left((1+B) + \sqrt{3B^2 + 8B + 3 + 2\overline{\varphi}_k\left(B^2\right)}\right)$$

The following lemma presents the different properties of such functions and shows that these latter admit limit functions.

Lemma 1 Consider the Feigenbaum equation defined by $\Delta_{t+1} + B\Delta_{t-1} = 2\mu\,\Delta_t + 2\,\Delta_t^2$ for all $t \in N$ where μ and B are two real parameters with $|B| < 1$ Feigenbaum. The paths of critical functions of the Jacobian matrix $(\varphi_k)_{k\in N}$ and $(\overline{\varphi}_k)_{k\in N}$ satisfy the following properties:

1) For all $k \geq 1$ and non zero *B*, we have $\varphi_k(1/|B|) = 1/|B|\varphi_k(|B|)$ and $\overline{\varphi}_k(1/|B|) = 1/|B|\overline{\varphi}_k(|B|)$.

2) φ_k and $\overline{\varphi}_k$ are defined and continuous on R for all $k \geq 1$.

3) For all $k \geq 1$, φ_k and $\overline{\varphi}_k$ have respectively the following asymptotes:

   When *B* tends toward $-\infty$, $\varphi_k(B) \approx \frac{1}{2}\left(1 + \sqrt{5 - 2\varphi_k(0)}\right)B$, when *B* tends toward $+\infty$, $\varphi_k(B) \approx \frac{1}{2}\left(1 - \sqrt{5 - 2\varphi_k(0)}\right)B$.

When B tends toward $-\infty$, $\overline{\varphi}_k(B) \approx \frac{1}{2}\left(1-\sqrt{3+2\overline{\varphi}_k(0)}\right)B$, when B tends toward $+\infty$,

$\overline{\varphi}_k(B) \approx \frac{1}{2}\left(1+\sqrt{3+2\overline{\varphi}_k(0)}\right)B$.

4) In *R*, $(\varphi_k)_{k\in N}$ and $(\overline{\varphi}_k)_{k\in N}$ converge to the continuous functions φ_∞ and $\overline{\varphi}_\infty$ which respectively satisfy:

$$\varphi_\infty\left(B^2\right) = -2\varphi_\infty^2(B) + 2(1+B)\varphi_\infty(B) + 2B^2 + 3B + 2 \text{ for all } k \in N$$

and $$\overline{\varphi}_\infty\left(B^2\right) = +2\overline{\varphi}_\infty^2(B) - 2(1+B)\overline{\varphi}_\infty(B) - B^2 - 3B - 1 \text{ for all } k \in N$$

Proof see Appendix B.

The critical function characterises the successive values and the limit-function of accumulation. It is useful to present these results in diagrammatic form.

[Insert Figure 2]

The intersection between the critical functions and the axis $B = 0$ corresponds with the critical values of the logistic equation. The intersection between the critical functions and the axis $B = -1$ or $B = 1$ corresponds with the critical values of the Hamiltonian case (Figure 2 does not allow us to distinguish the different functions beyond φ_2, that corresponds to the occurrence of an 8-cycle).

Figure 3 shows the mirror-critical functions obtained by mirror-symmetry. These functions take positive values on the chosen interval and are subject to the same analysis that we described above.

[Insert Figure 3]

3.2- Application of the technique in the case of a uni-dimensional dynamics

The model of Matsuyama (1999) presents a simple growth model which captures at the same time the main features of a Solow growth model and of a Romer growth model. Matsuyama is able to explain the rise of growth in an economy thanks to two features: innovation and investment.

In the case of a Romer regime, the dynamics of the model is:

$$k_t = G\,k_{t-1}/\left(1+\theta\left(k_{t-1}-1\right)\right) \text{ when } k_{t-1} \geq 1$$

where k measures the balance between the two engines of growth, i.e. capital accumulation and innovation, G measures the level of growth in the economy and θ is related to the monopoly power of the innovator.

The stationary equilibrium is $k^* = 1 + (G-1)/\theta$ and the Jacobian matrix at this stationary equilibrium is: $\begin{pmatrix} (1-\theta)/G & 0 \\ 0 & 0 \end{pmatrix}$

The trace is given by $(1-\theta)/G$ and the determinant is 0. We can then deduce the coefficients of the related Feigenbaum equation: $\mu = \text{Trace}/2 = (1-\theta)/(2G)$ and $B = 0$.

Let us notice that we are in the case of the logistic equation since $B = 0$:

$$\Delta_{t+1} = G(1-\theta)\Delta_t + 2\Delta_t^2$$

By letting $x_t = -\Delta_t/\mu$, we find the logistic equation: $x_{t+1} = 2\mu\, x_t\,(1-x_t)$.

From paragraph 2.1 (and by considering the case $B = 0$ depicted in Figure 2), the first bifurcation arises when $\mu_0 = -1/2$, i.e. $G = \theta - 1$. This is exactly the result found by

Matsuyama. The second bifurcation arises when $G = 1/(2\mu_1)(\theta - 1)$. According to the assumption on G and θ, this is impossible to reach, therefore this bifurcation cannot arise.

Thanks to this simple variable transformation, we are therefore able to show the impossibility of 3-period cycle in the model in a quicker way.

3.3- Application of the technique in the case of a multi-dimensional dynamics

We are going to apply the method in the well-known Kaldor's model[1] (1957). As is above pointed, we assume a constant Jacobian in order to keep a good approximation of the bifurcation parameters. For that purpose, we are going to provide a sufficiently generic investment function.

First, let us recall the basic framework of the Kaldor model. The macro-economic variables and parameters are as follows : Y_t is the production at the time period t, K_t the capital stock, I_t the investment, S_t the supply of goods, α the marginal propensity to consume, μ the marginal propensity to save and δ the rate of depreciation of the capital stock.

We look for an investment function compatible with the traditional assumptions of the Kaldor model.

[1] Note that some works have been made dealing with the chaotic dynamic of this model (see Dana and Malgrange (1984)) and Lorenz (1991)). However, their approaches are essentially based on numerical simulations and are not concerned with such an analytical approach as we developed in the first part of the paper.

A1: The saving function linearly depends on the production *Y* and does not depend on the capital stock, i.e. $S_Y = \mu$ where $0 < \mu < 1$.

A2: The investment function I is increasing in *Y*, i.e. $I_Y > 0$.

A3: The investment function I is decreasing in *K*, i.e. $I_K < 0$.

A4: There exists a production equilibrium $Y*$ such that:

$$\begin{aligned} I_{Y,Y} &> 0 \quad if \quad Y < Y* \\ I_{Y,Y} &< 0 \quad if \quad Y > Y* \end{aligned}$$

The dynamics (S_K) in the Kaldor model is defined as follows:

$$\begin{cases} Y_{t+1} - Y_t = \alpha\left(I\left(Y_t, K_t\right) - S\left(Y_t\right)\right) = \alpha I\left(Y_t, K_t\right) + \left(1 - \alpha\mu\right)Y_t \\ K_{t+1} - K_t = I\left(Y_t, K_t\right) - \delta K_t = I\left(Y_t, K_t\right) + \left(1 - \delta\right)K_t \end{cases}$$

The Jacobian matrix *J* of S_K at the stationary point is

$$J = \begin{pmatrix} \alpha I_Y + \left(1 - \alpha\mu\right) & \alpha I_K \\ I_Y & I_K + \left(1 - \delta\right) \end{pmatrix}$$

Its determinant is: $\mathrm{Det}\left(J\right) = \left(1 - \alpha\mu\right)I_K + \alpha\left(1 - \delta\right)I_Y + \left(1 - \delta\right)\left(1 - \alpha\mu\right)$.

By denoting $a = \left(1 - \alpha\mu\right)$ and $b = \alpha\left(1 - \delta\right)$, we have the following result.

Proposition Let an investment function $I\left(Y, K\right) = F\left(aY - bK + c\right) + dY + eK + f$, where $c \geq 0$, $d \geq 0, e \leq 0$ and $F: R \rightarrow R$ is a continuous and differentiable function such that:

i) $F\left(0\right) = 0$

ii) $F'\left(x\right) > 0$ where $x = aY - bK + c$

iii) $F''\left(0\right) = 0$, $F''\left(x\right) > 0$ *if* $x < 0$, $F''\left(x\right) < 0$ *if* $x > 0$.

1) The stationary equilibrium is $(Y^*, K^*) = (\delta c/(a\delta - \mu b), \mu c/(a\delta - \mu b))$

2) S_K (with the above investment function) satisfies A2-A4.

3) The determinant of the Jacobian matrix of S_K is constant, i.e. $\mathrm{Det}(J) = \alpha(1-\delta)d + (1-\alpha\mu)e + (1-\delta)(1-\alpha\mu)$.

As an example, we can specify an extension of the logistic function defined by: $e^x/(e^x+1)$. Then, the investment function takes the form:

$$I(Y,K) = \frac{e^{aY-bK+c}}{e^{aY-bK+c}+1} + dY + eK + f$$

4- Generalisation of Helleman's Method

In this section we are going to present a technique which allows to find the occurrence of a cascade of bifurcation in the case of a non constant Jacobian matrix at the steady state. Consider the following Feigenbaum equation:

$$\Delta_{t+1} + B\Delta_{t-1} = aB\Delta_t + 2\,\Delta_t^2 \text{ for all } t \in N$$

In this case we replace the trace of the Jacobian matrix issued from (1) by a linear function, i.e. aB. We are then able to establish the following definition of the critical value of the Jacobian matrix of the above Feigenbaum equation at the stationary point.

Definition 2 Consider the Feigenbaum equation defined by $\Delta_{t+1} + B\Delta_{t-1} = aB\Delta_t + 2\,\Delta_t^2$ where a is a positive real parameter and B a real parameter such as $|B| < 1$, $(\varphi_k)_{k \in N}$ and $(\overline{\varphi}_k)_{k \in N}$ the paths of critical functions of bifurcation.

The paths of critical values of the Jacobian matrix at the stationary point, $(B_k)_{k \in N}$ and $(\overline{B}_k)_{k \in N}$ are defined by the relationships $\varphi_k(B_k) = (a/2)B_k$ and $\overline{\varphi}_k(\overline{B}_k) = (a/2)\overline{B}_k$.

These critical values correspond to the transitions of the system and characterise the qualitative transformation of the dynamics. The usual approach initiated by Helleman assumes the fixity of the Jacobian matrix; in our case, it varies. By studying the qualitative transformations of the dynamics in terms of successive values of the Jacobian matrix, we can establish the following properties.

Lemma 2 Consider the Feigenbaum equation defined by $\Delta_{t+1} + B\Delta_{t-1} = aB\Delta_t + 2\,\Delta_t^2$ for $t \in N$, where a is a real parameter and B a real parameter such as $|B| < 1$, $(\varphi_k)_{k \in N}$ and $(\overline{\varphi}_k)_{k \in N}$ the paths of critical functions of bifurcation.

1) For all integer k, if $a > \frac{1}{2}\left(1 + \sqrt{5 - 2\varphi_\infty(0)}\right)$, then $(B_k)_{k \in N}$ exists and is composed of negative terms.
2) For all integer k, if $a > \frac{1}{2}\left(1 + \sqrt{3 + 2\overline{\varphi}_\infty(0)}\right)$, then $(\overline{B}_k)_{k \in N}$ exists and is composed of positive terms.

3) If $\left(B_k\right)_{k \in N}$ is defined, then it is decreasing.

4) If $\left(\overline{B}_k\right)_{k \in N}$ is defined, then it is increasing.

5) If $a > 2\sqrt{\left(\sqrt{65}+1\right)/8}$, then $\left(B_k\right)_{k \in N}$ belongs to $\left(-1, 0\right]$.

6) If $a > 1 + \sqrt[4]{65}$, then $\left(\overline{B}_k\right)_{k \in N}$ belongs to $\left[0, 1\right)$.

Proof See Appendix D.

[Insert Figure 4]

Figure 4 shows the asymptotic branches of each critical function when the modulus of *B* tends towards infinity.

[Insert Figure 5]

Figure 5 gives a closer look at Figure 5. Here we are able to distinguish the behaviour of critical functions on $\left[-1,1\right]$ in terms of two values of *a* (one 'high' and greater than 2, one 'low', less than 1). The intersection between the line *y* and the critical functions shows the emergence of cycles. In the case of a low value of *a*, the 8-cycle appears in a point $B*$ less than –1.

Since the Jacobian matrix is locally constant or varies a little, its value at the equilibrium of period 4 is close to $\left|B*\right|^4$ and all the following orbits which successively appear are unstable. If *a* is quite 'high' (greater than 2), all cycles are stable with regard to the previous argument.

Let us finally first make two remarks.

First, for a given value of *a*, the range of possible dynamics is limited. Indeed, when the parameter belongs to a given interval, we obtain some qualitative information on the dynamics.

Second, not all variations of the Jacobian matrix does not necessarily imply that dynamics go towards complexity.

5- Economic Application of the Generalised Method

Médio (1992) analyses numerically the backward dynamics of Reichlin's model (1986) in the case of constant absolute risk aversion. For this purpose, he postulates the following utility: $u(c) = C - r\exp(\text{- c})$, where r is the risk aversion coefficient and C a positive constant, and lets $U(c) = u'(c)c = r\,c\exp(-c)$. He specifies the following disutility function $v(l) = 1/\beta\, l^{\beta}$ and obtains $V(l) = l\,v'(l) = l^{\beta}$.

The following system of equations describes the backward dynamics of Reichlin's model:

$$\begin{cases} l_t = g(c_{t+1}) \\ c_t = g(c_{t+1})/a_0 - l_{t+1}/(a_0\, a) \end{cases} \quad \forall t \in N$$

where $g(c) = V^{-1}[U(c)] = (r\ c\exp(-c))^{1/\beta}$

Médio assumes $a_0 = 1$. Given the initial conditions for l and c, this system describes the dynamics "in the past". Then let us quote Médio (1992, p. 223) "adopting (and keeping in mind) the convention that successive (integer) values of the independent variable $t = 1,2,3\ldots$ are now taken to correspond to points on the negative half of the real line", we can rewrite this system as the system (S2):

$$\begin{cases} l_{t+1} = g(c_t) \\ c_{t+1} = g(c_t)/a_0 - l_t/a \end{cases} \quad \forall t \in N \tag{S2}$$

which is well defined for $c_t \in [0,+\infty)$ and $l_t \in [0,+\infty)$, $\forall t \in N^*$.

Let us study the dynamics of (S2) around the stationary point (l^*, c^*). (From now on we are going to denote the stationary point $(l^*(a), c^*(a))$ as (l^*, c^*) for ease of presentation.) Denote $l_t = l^* + \delta_t$ and $c_t = c^* + \varepsilon_t$, for all $t \in N$. Rewrite (S2) around the stationary point and call *A*, the Jacobian matrix of (S2) in this point:

$$A = \begin{pmatrix} 0 & g'(c^*) \\ -1/a & g'(c^*) \end{pmatrix}$$

By following the steps described in section 2 (see Appendix E), we can rewrite (S2) as the following Feigenbaum equation:

$$\tilde{\Delta}_{t+1} + B(a)\tilde{\Delta}_{t-1} = 2\mu(A)\tilde{\Delta}_t + 2\,\tilde{\Delta}_t^2 \text{ for all } t \in N$$

where $B(a) = 1/\mathrm{Det}(A) = a/g'(c^*) < 1$ and $\mu(A) = \mathrm{Tr}(A)/(2\,\mathrm{Det}(A)) = a/2$.

In the vicinity of the stationary point this Feigenbaum equation is equivalent to:

$$\Delta_{t+1} + B\,\Delta_{t-1} = a\,B\,\Delta_t + 2\Delta_t^2$$

with the following relationship (see Médio, 1992: p.226):

$$r = \exp(c^*)c^{*\beta-1}(a/(a-1))^{\beta},$$

$$\mathrm{Tr}[\mathrm{J}[\phi(c^*, l^*)]] = a/(\beta\,(a-1))(1-c^*)$$

and $\quad \mathrm{Det}[\mathrm{J}[\phi(c^*, l^*)]] = 1/(\beta\,(a-1))(1-c^*)$.

The numerical experiments of the backward dynamics made by Médio give similar results to those obtained from the Feigenbaum equation. Indeed, from the latter equations, it is possible to find a path $(r_k)_{k\in N}$ of values of bifurcation. We can then establish the following relationship:

$$r_k = [\exp(1 - B_k\, \beta\, (a-1))](1 - B_k\, \beta\, (a-1))^{\beta - 1}(a/(a-1))^{\beta} \text{ for all } k \in N$$

where $(B_k)_{k\in N}$ is the path of critical values linked to the Feigenbaum equation. The numerical evaluation of critical values is difficult to obtain because the equations we have to solve are transcendent. Let us consider the following polynomial:

$$P_B(\mu) = -2\mu^2 + 2(B+1)\mu + 2B^2 + 3B + 2$$

We can use the definition of critical values of bifurcation to show that the critical value associated with the occurrence of a 2^{k+1} period-cycle is one of the solutions of the equation of 2^k-degree: $\mathop{\mathrm{O}}\limits_{k=0}^{n} P_{B^{2^k}}(aB/2) = -\left(1 + B^{2^k}\right)/2$

where O is the composition operator. Having a numerical analysis of the critical values beyond the fifth is hard. We have therefore tried to evaluate the first six values.

[Insert Table 1]

All critical values of B belong to $[-1,0]$ when $a > 2\sqrt{\sqrt{65}+1/8} \approx 2.12864$. When a is close to 2.12864 by higher values, the path of critical values converges to a value close to –1 (see Figure 6 and Table 1).

[Insert Figure 6]

Consider now the case of a “high” productivity (see Figure 7).

[Insert Figure 7]

When $a > 2.12864$ (or $a = 5$), our numerical analysis looks like that of Médio. However the values of *r* are relatively different from those of Feigenbaum equation. For example, the simulations show that the entrance in the chaotic area appears when *r* is ‘above 105’. In the case of the Feigenbaum equation, this value is close to 146.505. This rough estimate could be attributed to the specification of the exponential function and/or the negligence of the terms of three-degree. This can be seen as one of the disadvantages of the method. Remember that the simplicity of this technique results from approximation and the estimates we obtain give us an idea of what to expect. Indeed in order to obtain the Feigenbaum equation, it is necessary to approximate the difference equation of the original model by a Taylor series expansion of order two (the terms of order greater or equal to three are eliminated). The difference between the critical values (these latter being observed by numerical examples) corresponding with the dynamic transitions of the original system and those obtained by the Feigenbaum equation can therefore be significant, but we are still able to give a qualitative study around the stationary point.

6- Conclusion

The purpose of our paper was twofold. First we wanted to apply the renormalization procedure in macrodynamics in order to show its tractability and its advantages. The economic models we used dealt with constant Jacobian. Second since this assumption can be considered too restrictive, we generalised this procedure to non constant cases and applied it to a well-known economic model.

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Appendix

Appendix A- Feigenbaum Equation and Renormalization

Let us consider the Feigenbaum equation defined as follows:

$$\Delta_{t+1} + B\Delta_{t-1} = 2\mu\,\Delta_t + 2\,\Delta_t^2 \text{ for all } t \in N \tag{A.1}$$

and let us study its dynamics with respect to μ.

Stability of the dynamics

Let $\varepsilon_t = \Delta_{t-1}$ and rewrite (A.1) in the two-dimensional space, we then obtain for all $t \in N$

$$\begin{pmatrix} \Delta_{t+1} \\ \varepsilon_{t+1} \end{pmatrix} = \begin{pmatrix} 2\mu\,\Delta_t + 2\Delta_t - B\varepsilon_t \\ \Delta_t \end{pmatrix}$$

The point $(0,0)$ is a stationary equilibrium. The Jacobian matrix at this point is $\begin{pmatrix} 2\mu & -B \\ 1 & 0 \end{pmatrix}$. Hartman's theorem shows the equilibrium loses its stability when the eigenvalues of the Jacobian matrix cross the unit circle. It is then trivial to prove that the equilibrium becomes unstable when $\mu < -(1+B)/2$.

Mirror-symmetry

Let us find another critical value by introducing the following linear transformation for $t \in N$: $\overline{\Delta}_t = \Delta_t - \mu + (1+B)/2$.

Substituting $\overline{\Delta}_t$ into (A.1) yields: $\overline{\Delta}_{t+1} + B\overline{\Delta}_{t-1} = 2\overline{\mu}\,\overline{\Delta}_t + 2\,\overline{\Delta}_t^2$ where $\overline{\mu} = 1 + B - \mu$.

The point $(0,0)$ is also an equilibrium of the above equation. It becomes unstable when $\overline{\mu} < -(1+B)/2$. This condition can be rewritten as $\mu > 3(1+B)/2$.

Two-period cycle

(A.1) has a two-period path defined by:

$\Delta_t = -1/2\left((1+B)/2+\mu\right)+(-1)^t/2\left(\sqrt{((1+B)/2+\mu)(-3(1+B)/2+\mu)}\right)$ for all $t \in N$

When $\mu < -(1+B)/2$ and $\mu > 3(1+B)/2$, the equilibrium of (A.1) becomes unstable and a 2-period cycle appears.

Renormalization

Since $|B| < 1$, the system associated with (A.1) is dissipative. Let $(\xi_t)_{t\in N}$ the 2-period path obtained from (A.1), with $\begin{cases}\xi_{2t+1} = \xi_1 \\ \xi_{2t} = \xi_0\end{cases}$ for all $t \in N$.

Let $\Delta_t = \delta_t + \xi_t$, where δ_t measures the distance between the path and the 2-period cycle.

$$\delta_{t+1} + B\delta_{t-1} = (2\mu + 4\xi_t)\delta_t + 2\,\delta_t^2 \text{ for all } t \in N \tag{A.2}$$

Write (A.2) at time $t = 2\tau + 1$, $t = 2\tau$ and $t = 2\tau - 1$.

$$\delta_{2\tau+2} + B\delta_{2\tau} = 2\,\delta_{\tau+1}^2 + (2\mu + 4\xi_{2\tau+1})\delta_{2\tau+1} \tag{A.3}$$

$$\delta_{2\tau} + B\delta_{2\tau-2} = 2\,\delta_{\tau-1}^2 + (2\mu + 4\xi_{2\tau-1})\delta_{2\tau-1} \tag{A.4}$$

$$\delta_{2\tau+1} + B\delta_{2\tau-1} = 2\,\delta_{\tau}^2 + (2\mu + 4\xi_{2\tau+1})\delta_{2\tau} \tag{A.5}$$

Let $d = 2\mu + 4\xi_{2\tau}$ and $e = 2\mu + 4\xi_{2\tau-1}$

Evaluating $(\mathrm{A.3}) + B(\mathrm{A.4}) + e(\mathrm{A.5})$ yields:

$$\delta_{2\tau+2} + B^2\,\delta_{2\tau-2} = 2\left(\delta_{2\tau+1}^2 + B\delta_{2\tau-1}^2\right) + 2e\delta_{2\tau}^2 + \delta_{2\tau}(-2B + e\,d) \tag{A.6}$$

by letting $B' = B^2$ and $\mu' = d\,e/2 - B$, (A.6) becomes:

$$\delta_{2\tau+2} + B'\,\delta_{2\tau-2} = 2\left(\delta_{2\tau+1}^2 + B\delta_{2\tau-1}^2\right) + 2e\delta_{2\tau}^2 + 2\,\delta_{2\tau}(-B + d\,e/2) \tag{A.7}$$

Let us evaluate the terms between brackets in another way by letting $r(\delta) = \delta_{2\tau+1}/\delta_{2\tau-1}$

$$\delta_{2\tau+1}^2 + B\delta_{2\tau-1}^2 = 2\delta_{2\tau-1}^2\left(\delta_{2\tau+1}^2/\delta_{2\tau-1}^2 + B\right) = 2\delta_{2\tau-1}^2\left(r^2(\delta) + B\right) \tag{A.8}$$

Substituting $r(\delta)$ into (A.5) and neglecting the terms of degree 2 yields:

$$\delta_{2\tau-1}(B + r(\delta)) = d\delta_{2\tau} \iff d\,\delta_{2\tau}/(B + r(\delta)) = \delta_{2\tau-1}$$

substituting this expression into (A.8) yields:

$$2\left(\delta_{2\tau+1}^2 + B\delta_{2\tau-1}^2\right) = 2\,d^2\,\delta_{2\tau}^2/(r(\delta) + B)^2\left(r^2(\delta) + B\right)$$

Since we are located around the 2-period cycle, $r = 1$ therefore:

$$2\left(\delta_{2\tau+1}^2 + B\delta_{2\tau-1}^2\right) = 2\,d^2\,\delta_{2\tau}^2/(1 + B)$$

Substituting this expression into (A.7) yields:

$$\delta_{2\tau+2} + B'\,\delta_{2\tau-2} = 2\left(d^2/(1+B) + e\right)\delta_{2\tau}^2 + 2\,\delta_{2\tau}(-B + d\,e/2)$$

Let $\alpha = e + d^2/(1+B)$ and $\mu' = d\,e/2 - B$ and multiply the previous expression by α, we then obtain:

$$\alpha\,\delta_{2\tau+2} + B'\,\delta_{2\tau-2} = 2\alpha^2\,\delta_{2\tau}^2 + 2\,\mu'\alpha\,\delta_{2\tau}$$

Let $\Delta_\tau = \alpha\,\delta_{2\tau}$ then we obtain: $\Delta_{\tau+1} + B'\Delta_{\tau-1} = 2\,\mu'\Delta_t + 2\,\Delta_t^2$

Since $\xi_t^{1,2} = a + (-1)^t |b| = -1/2\left((1+B)/2 + \mu\right) + (-1)^t /2 \left(\sqrt{((1+B)/2+\mu)(-3(1+B)/2+\mu)}\right)$

We can then evaluate μ' from μ

$$\mu' = 1/2(2\mu + 4\xi_{t+1}) - B$$

$$\mu' = 1/2\left(2\mu + 4(a+b)\right)\left(2\mu + 4(a-b)\right) - B$$

$$\mu' = -2\mu^2 + 2\mu(1+B) + 2B^2 + 3B + 2$$

Since $\mu'\left(B^2\right) = 1 + B^2 - \overline{\mu}'\left(B^2\right)$, we have:

$$\overline{\mu}'\left(B^2\right) = 2\overline{\mu}^2(B) - 2\ (1+B)\overline{\mu}(B) - B^2 - 3B - 1 \qquad \text{(A.9)}$$

It is usually said that (A.9) is obtained after a renormalization procedure.

Any 2^k-period cycle of this equation is a 2^k-period cycle for (A.9). We now need to provide a recurrent path defined with respect to μ for which the 2^k-period cycle loses its stability while a stable 2^{k+1}-period cycle appears.

For any real value *B*, consider the equations obtained from the renormalization procedure:

$$\Delta^{(0)}_{\tau_0+1} + B^{2^0}\ \Delta^{(0)}_{\tau_0-1} = 2\mu^{(0)}\Delta^{(0)}_{\tau_0} + 2\ \Delta^{(0)^2}_{\tau_0} \text{ for all } \tau_0 \in 2^0 N \qquad \text{(A.10.0)}$$

(…)

$$\Delta^{(k-1)}_{\tau_{k-1}+1} + B^{2^{k-1}}\ \Delta^{(k-1)}_{\tau_{k-1}-1} = 2\mu^{(k-1)}\Delta^{(k-1)}_{\tau_{k-1}} + 2\ \Delta^{(k-1)^2}_{\tau_{k-1}} \text{ for all } \tau_{k-1} \in 2^{k-1} N \qquad \text{(A.10.}k\text{-1)}$$

(A.10.0) is equal to (A.9) and each of the following equations is obtained from the renormalization procedure. Any variable $\Delta^{(j)}_{\tau_j}$, for $j = 1,\ldots,k-1$ is obtained after *j* renormalization procedures and the parameters $\mu^{(j)}$, for $j = 1,\ldots,k-1$ are defined by:

$$\mu^{(j)} = -2\mu^{(j-1)2} + 2\left(1 + B^{2^{(j-1)}}\right)\mu^{(j-1)} + 2B^{4^{(j)}} + 3B^{2^{(j-1)}} + 2$$

Let $\mu_0\left(B^{2^k}\right)$ be the critical value of the parameter $\mu^{(k-1)}$ which corresponds to the loss of stability of an equilibrium while a 2-period cycle appears. This critical value is obtained from:

$$\mu_0\left(B^{2^{k-1}}\right) = -2\mu_1^{\,2}\left(B^{2^{k-2}}\right) + 2\left(1 + B^{2^{k-2}}\right)\mu_1\left(B^{4^{k-2}}\right) + 2B^2 + 3B^{2^{k-2}} + 2$$

where $\mu_1\left(B^{2^{k-1}}\right)$ is the value of the parameter $\mu^{(k-2)}$ for which a 4-period cycle appears in (A.10.*k*-2).

By applying this idea to (A.10.*k*-1) and by coming back step by step to the first equation we obtain:

$$\mu_k\left(B^2\right) = -2\mu_{k+1}^{\,2}(B) + 2(1+B)\mu_{k+1}(B) + 2B^2 + 3B + 2$$

Similarly the mirror-symmetry leads to:

$$\overline{\mu}_k\left(B^2\right) = +2\overline{\mu}_{k+1}^{\,2}(B) - 2(1+B)\overline{\mu}_{k+1}(B) - B^2 - 3B - 1$$

Therefore it is possible to provide a path of critical value of μ which corresponds to a critical transformation of the dynamics.

Appendix B: Proof of Lemma 1

1) Since $\varphi_0(B) = -(1+B)/2$ and $\overline{\varphi}_0(B) = 3(1+B)/2$ and by writing the recurrence relationship which describes the critical functions, we deduce that φ_1 and $\overline{\varphi}_1$ satisfy

this relationship. It is then trivial to prove that if the relationship holds for *k*, then it holds for $k+1$. It is therefore satisfied for all positive *k*.

2) Let us first show that φ_k is defined on $(-1,1)$. Since we have $\varphi_0(B) = -(1+B)/2$, we deduce $\varphi_0(B^2) < 0$ for all B. Since $5B^2 + 8B + 5 \geq 0$ for all $B \in R$ and $\varphi_0(B^2) < 0$, φ is non positive, well defined on R and continuous. By recurrence we deduce that if φ_k is defined and continuous over R, then $\varphi_{k+1}(B)$ is defined and continuous over R.

3) We know $k \geq 1$, $\varphi_k(|B|) = |B|\, \varphi_k(1/|B|)$ (see 1). Recall $\varphi_{k+1}(B) = \frac{1}{2}\left((1+B) - \sqrt{5B^2 + 8B + 5 - 2\varphi_k(B^2)}\right)$, when B tends toward $-\infty$, and $|B| = -B$.

Hence, $\dfrac{\varphi_{k+1}(B)}{B} = \dfrac{1}{2}\left(\dfrac{1+B}{B} + \dfrac{B}{B}\sqrt{5 + 8\dfrac{1}{B} + 5\dfrac{1}{B^2} - \dfrac{\varphi_k(B^2)}{B^2}}\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{B} + 1 + \sqrt{5 + 8\dfrac{1}{B} + 5\dfrac{1}{B^2} - \varphi_k\left(\dfrac{1}{B^2}\right)}\right)$. By taking the limit yields $\dfrac{\varphi_{k+1}(B)}{B} \approx \dfrac{1}{2}\left(1 + \sqrt{5 - \varphi_k(0)}\right)$. The others results can be obtained in a similar way.

4) For any real B, let us denote $\varphi_\infty(B) = \underset{k\to\infty}{\mathrm{Lim}}\, \varphi_k(B)$ and $\overline{\varphi}_\infty(B) = \underset{k\to\infty}{\mathrm{Lim}}\, \overline{\varphi}_k(B)$. The first step is to prove that that $(\varphi_k)_{k\in N}$ is a Cauchy sequence on the interval $[-1,1]$. Consider:

$$\varphi_k(B^2) = -2\varphi_{k+1}^2(B) + 2(1+B)\varphi_{k+1}(B) + 2B^2 + 3B + 2 \text{ for all } k \in N$$

By substituting into this equation the following variable transformation:

$\psi_k(B) = 2\varphi_k(B) - (1+B)$ for all $k \in N$ we obtain:

$$\psi_k\left(B^2\right) = -\psi_{k+1}^2(B) + 4(1+B)^2$$

Let us study now the uniform convergence of this sequence on $[-1,1]$. For all *n* and *p*, we have:

$$\left|\psi_{n+p}(B) - \psi_n(B)\right| = \left|\frac{\psi_{n+p-1}\left(B^2\right) - \psi_{n-1}\left(B^2\right)}{\psi_{n+p-1}(B) + \psi_{n-1}(B)}\right|$$

However, from 2) we have:

$$\operatorname*{Inf}_{B \in [-1,1]} \left|\psi_{n+p-1}(B) + \psi_{n-1}(B)\right| = \gamma > \operatorname*{Inf}_{B \in [-1,1]} \left|2\psi_0(B)\right| = 0$$

Moreover from 2) we have:

$$\left|\psi_{n+p-1}\left(B^2\right) - \psi_{n-1}\left(B^2\right)\right| = \left|\frac{\psi_{n+p-1}\left(B^4\right) - \psi_{n-1}\left(B^4\right)}{\psi_{n+p-1}\left(B^2\right) + \psi_{n-1}\left(B^2\right)}\right|$$

Furthermore for n, p and $k > 0$, we obtain from 2):

$$\operatorname*{Inf}_{B \in [-1,1]} \left|\psi_{n+p-k}\left(B^{2k}\right) + \psi_{n-1}\left(B^{2k}\right)\right| > \operatorname*{Inf}_{B \in [-1,1]} \left|2\psi_0\left(B^{2k}\right)\right| = 3$$

We then deduce:

$$\operatorname*{Sup}_{B \in [-1,1]} \left|\psi_{n+p}(B) - \psi_n(B)\right| < \frac{1}{\gamma}\left(\frac{1}{3}\right)^{n-1} \operatorname*{Sup}_{B \in [-1,1]} \left|\psi_p\left(B^{2n}\right) + \psi_0\left(B^{2n}\right)\right|$$

Since for all $k \in N$, ψ_k is a continuous function. Then for all $B \in [-1,1]$:

$$\lim_{n \to \infty} \left|\psi_p\left(B^{2n}\right)\right| = \left|\psi_p(0)\right| < \left|\psi_\infty(0)\right| = \left|2\mu_\infty - 1\right| = \left(3 + \sqrt{17}\right)/2$$

Recall: $\left|\psi_p(-1)\right| < \left|\psi_\infty(-1)\right| = \left|2\varphi_\infty(-1)\right| = 2\sqrt{\left(1+\sqrt{65}\right)/8}$

and $\quad \left|\psi_p(1)\right| < \left|\psi_\infty(1)\right| = \left|2\varphi_\infty(1) - 2\right| = \left(1+\sqrt{65}\right)/2$

Thus $\lim\limits_{n\to\infty}\left(\sup\limits_{B\in[-1,1]}\left|\psi_{n+p}(B) - \psi_n(B)\right|\right) < \lim\limits_{n\to\infty}\left[\frac{1}{\gamma}\left(\frac{1}{3}\right)^{n-1}\sup\limits_{B\in[-1,1]}\left|\psi_p\left(B^{2n}\right) + \psi_0\left(B^{2n}\right)\right|\right]$

$$\lim\limits_{n\to\infty}\left(\sup\limits_{B\in[-1,1]}\left|\psi_{n+p}(B) - \psi_n(B)\right|\right) < \lim\limits_{n\to\infty}\left[\frac{1}{\gamma}\left(\frac{1}{3}\right)^{n-1}\right]\lim\limits_{n\to\infty}\left[\sup\limits_{B\in[-1,1]}\left|\psi_p\left(B^{2n}\right) + \psi_0\left(B^{2n}\right)\right|\right]$$

$$< \lim\limits_{n\to\infty}\left[\frac{1}{\gamma}\left(\frac{1}{3}\right)^{n-1}\right]\left(\frac{7+\sqrt{65}}{2}\right)$$

Thus for all $p \in N$, $\lim\limits_{n\to\infty}\left(\sup\limits_{B\in[-1,1]}\left|\psi_{n+p}(B) - \psi_n(B)\right|\right) = 0$.

Since the set of the continuous functions on the compact set $[-1,1]$ is a Banach set with respect to the uniform, the sequence $(\psi_k)_{k\in N}$ converges uniformly towards some continuous functions ψ_∞ on $[-1,1]$. Moreover, when $|B|>1$, $\psi_k(|B|) = |B|\,\psi_k\left(1/|B|\right)$ implies that when B>1, $\psi_k(B) = B\psi_k(1/B)$and the same result applies. Consequently, applying the same method to the function $\overline{\varphi}_k(B)$, we deduce that when $B \geq -1$, there exist two continuous functions over $[-1,+\infty]$denoted φ_∞ and $\overline{\varphi}_\infty$ such that:

$$\varphi_\infty\left(B^2\right) = -2\varphi_\infty^2(B) + 2(1+B)\varphi_\infty(B) + 2B^2 + 3B + 2 \text{ for all } k \in N$$

$$\overline{\varphi}_\infty\left(B^2\right) = +2\overline{\varphi}_\infty^2(B) - 2(1+B)\overline{\varphi}_\infty(B) - B^2 - 3B - 1 \text{ for all } k \in N$$

Now we need to extend this result to the case B<-1. By definition

$$\varphi_{k+1}(B) = \frac{1}{2}\left((1+B) - \sqrt{5B^2 + 8B + 5 - 2\varphi_k\left(B^2\right)}\right)$$

and $\qquad \overline{\varphi}_{k+1}(B)=\frac{1}{2}\left((1+B)+\sqrt{3B^2+8B+3+2\overline{\varphi}_k\left(B^2\right)}\right)$

Thus, when B<0 (a case including B<-1), then we obtain:

$$\lim_{k\to+\infty} \varphi_{k+1}(B)=\frac{1}{2}\left((1+B)-\sqrt{5B^2+8B+5-2\varphi_\infty\left(B^2\right)}\right)$$

$$\text{and} \lim_{k\to+\infty} \overline{\varphi}_{k+1}(B)=\frac{1}{2}\left((1+B)-\sqrt{3B^2+8B+3+2\overline{\varphi}_\infty\left(B^2\right)}\right)$$

Thus the functions $\varphi_\infty \ : B \to \lim_{k\to+\infty} \varphi_{k+1}(B)$ and $\overline{\varphi}_\infty \ : B \to \lim_{k\to+\infty} \overline{\varphi}_{k+1}(B)$ are continuous over R for any $k>0$.

Appendix C: Proof of the Proposition

1) At the stationary equilibrium, $I_{YY}(Y^*,K^*)=0$. Thus, $F''(aY^*-bK^*+c)=0$ $\Rightarrow aY^*-bK^*+c=0$. Moreover, from the equilibrium conditions we have $Y^*=\delta/\mu\, K^*$. Substituting these expressions in the equations of the model yields $(Y^*,K^*)=(\delta c/(a\delta-\mu b),\mu c/(a\delta-\mu b))$.

2) Since $F'(x)>0$, $F_Y = aF'(aY-bK+c)>0$. Since $d\geq 0$, $I_Y = aF'(aY-bK+c)+d>0$ and A2 holds.

   Moreover $F_K = -bF'(aY-bK+c)+e<0$ and since $e\leq 0$, A3 follows.

   Finally, $I_{YY} = a^2F''(aY-bK+c)$.

3) At the equilibrium $I_{YY} = a^2F''(aY^*-bK^*+c)=a^2F''(0)=0$, and we deduce the result. The determinant of the Jacobian is

$$\mathrm{Det}(J)=(1-\alpha\mu)I_K+\alpha(1-\delta)I_Y+(1-\delta)(1-\alpha\mu)=aI_K+bI_Y+(1-\delta)(1-\alpha\mu)$$

$$=b(aF'(aY-bK+c)+d)+a(-bF'(aY-bK+c)+e)+(1-\delta)(1-\alpha\mu)$$

$$=bd+ae+(1-\delta)(1-\alpha\mu)$$

$$=\alpha(1-\delta)d+(1-\alpha\mu)e+(1-\delta)(1-\alpha\mu)$$

Appendix D: Proof of Lemma 2

Results 1) and 2) are straightforward. Recall that $(\varphi_k(B))_{k\in N}$ and $(\overline{\varphi}_k(B))_{k\in N}$ are decreasing for any real B.

3) For all integer k, $\varphi_k(B_{k+1})>\varphi_{k+1}(B_{k+1})=a/2\,B_{k+1}$. Since $\varphi_k(0)<0$ and by applying the mean-value theorem, we deduce that there exists $B_k\in(B_{k+1},0)$ such that $\varphi_k(B_k)=a/2\,B_k$. Therefore, $B_{k+1}<B_k$ and 3) is shown.

4) By applying the method used to demonstrate 3), it is trivial to show 4).

5) It is trivial to show that $\varphi_\infty(-1)=\sqrt{\left(1+\sqrt{65}\right)/8}$, then 7) follows.

6) It is trivial to show that $\overline{\varphi}_\infty(1)=1+\sqrt[4]{65}$, then 8) follows.

Appendix E: Feigenbaum equation in Médio's model

By computing a Taylor expansion of degree two at the stationary point, (S2) becomes:

$$\begin{pmatrix}\delta_t\\ \varepsilon_t\end{pmatrix}=A\begin{pmatrix}\delta_{t+1}\\ \varepsilon_{t+1}\end{pmatrix}+\frac{1}{2}g''(c^*)\varepsilon_{t+1}^2$$

The characteristic equation of A is equal to:

$$\lambda^2-\mathrm{Tr}(A)\lambda+\mathrm{Det}(A)=0$$

$$\Leftrightarrow \qquad \lambda^2 - g'(c*)\lambda + g'(c*)/a = 0$$

where $\mathrm{Tr}(A)$ is the trace of matrix A and $\mathrm{Det}(A)$ its determinant.

The matrix A is diagonalizable if the eigenvalues of A are real and different, i.e. if the discriminant associated to the characteristic equation of A is strictly positive. We therefore have:

If $g'(c*) > (<) 0$, then $g'(c*) > (<) 4/a$.

Assume one of these two inequalities is verified. Make a variable transformation on the basis of eigenvectors. Denote (X_t, Y_t) the component of the system with respect to the basis of eigenvectors and λ_1 and λ_2, the different eigenvalues of the matrix *A*. We then obtain:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} + \begin{pmatrix} K_1 (X_{t+1} + Y_{t+1})^2 \\ K_2 (X_{t+1} + Y_{t+1})^2 \end{pmatrix} \tag{D.1}$$

where K_1 and K_2 are the coefficients evaluated from the transition matrix on the basis of eigenvectors. This former system a backward dynamics. We assume that $\lambda_1 \lambda_2 > 1$, i.e. the system is dissipative.

Let us assume $X_t/\lambda_1 + Y_t/\lambda_2 = X_{t+1} + Y_{t+1}$ and denote as $W_t = X_t - Y_t$ and $Z_t = X_t + Y_t$.

We can then deduce: $(Z_t + W_t)/(2\lambda_1) + (Z_t - W_t)/(2\lambda_2) = 2Z_{t+1}/2$

$$\Leftrightarrow \qquad Z_t(1/\lambda_1 + 1/\lambda_2) + W_t(1/\lambda_1 - 1/\lambda_2) = 2Z_{t+1}$$

$$\Leftrightarrow \qquad W_{t-1}(1/\lambda_1 - 1/\lambda_2) = 2Z_t - Z_{t-1}(1/\lambda_1 + 1/\lambda_2)$$

$$\Leftrightarrow \qquad W_{t-1} = \lambda_1 \lambda_2 / (\lambda_2 - \lambda_1)(2Z_t - Z_{t-1}(1/\lambda_1 + 1/\lambda_2))$$

The first co-ordinate of (D.1) is equal to $X_t = \lambda_1 X_{t+1} + K_1 (Z_{t+1})^2$

$$\Leftrightarrow \qquad W_{t-1} + Z_{t+1} = \lambda_1 / (\lambda_2 - \lambda_1)(2\lambda_2 Z_t - 2Z_{t-1})$$

and $\lambda_1 (W_t + Z_t) + 2K_1 Z_t^2 = \lambda_1/(\lambda_2 - \lambda_1)\left(-2\lambda_1 Z_t + 2\lambda_1 \lambda_2 Z_{t+1} + 2K_1(\lambda_2/\lambda_1 - 1)Z_t^2\right)$

Equating these two results enables us to find:

$$(\lambda_1 + \lambda_{21})Z_t + K_1(1 - \lambda_2/\lambda_1)Z_t^2 = Z_{t-1} + \lambda_1 \lambda_2 Z_{t+1} \qquad \text{(D.2)}$$

letting $K = (1 - \lambda_2/\lambda_1)K_1$, $J = \lambda_1 \lambda_2$ and $\eta = (\lambda_1 + \lambda_2)/2$

(D.2) $\Leftrightarrow Z_{t-1} + J\, Z_{t+1} = 2\eta\, Z_t + K\, Z_t^2$ (D.3)

by letting $Z_t = 2\, \Delta_t / K$, we have:

(D.3) $\Leftrightarrow \Delta_{t-1} + J\, \Delta_{t+1} = 2\eta\, \Delta_t + 2\Delta_t^2$ (D.4)

Let $\Delta_t = J\, \tilde{\Delta}_t$, $\mu = \eta / J$ and $B = 1/J$, we then obtain:

(D.4) $\Leftrightarrow \tilde{\Delta}_{t+1} + B\tilde{\Delta}_{t-1} = 2\mu\, \tilde{\Delta}_t + 2\tilde{\Delta}_t^2$

Figures

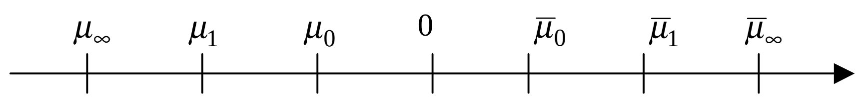

Figure 1

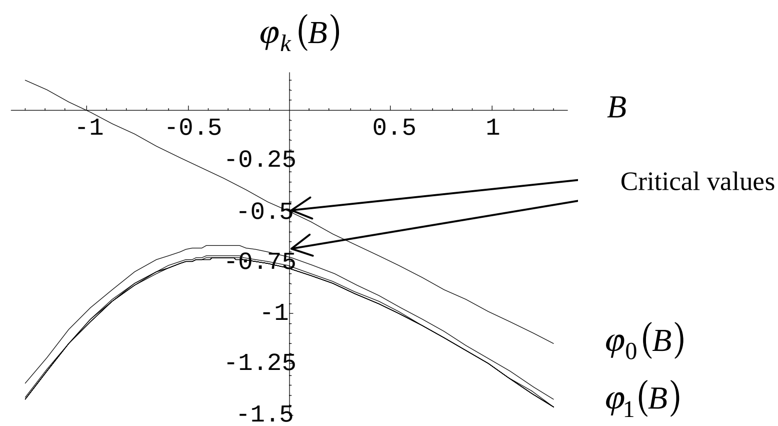

Figure 2

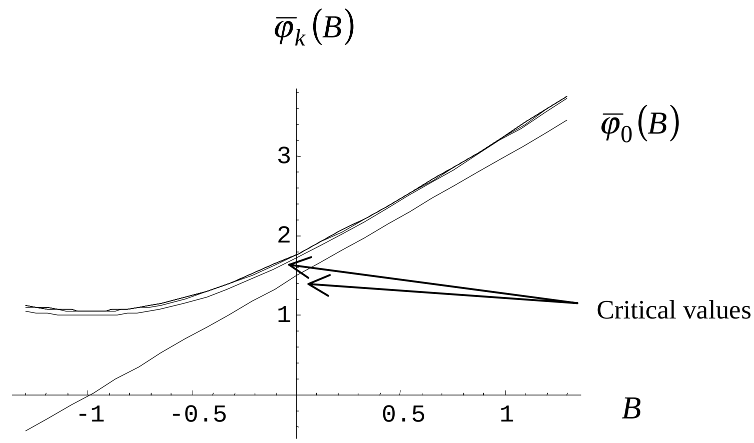

Figure 3

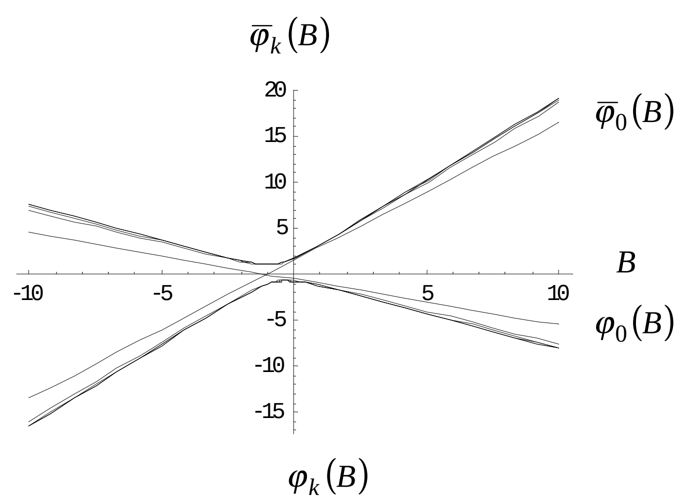

Figure 4

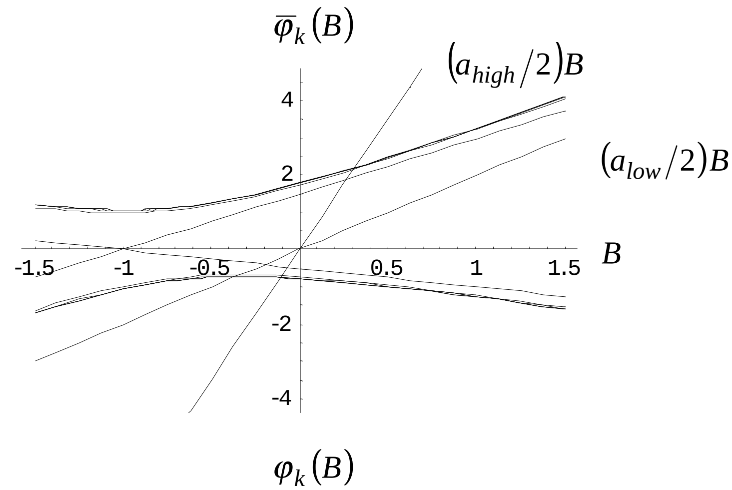

Figure 5

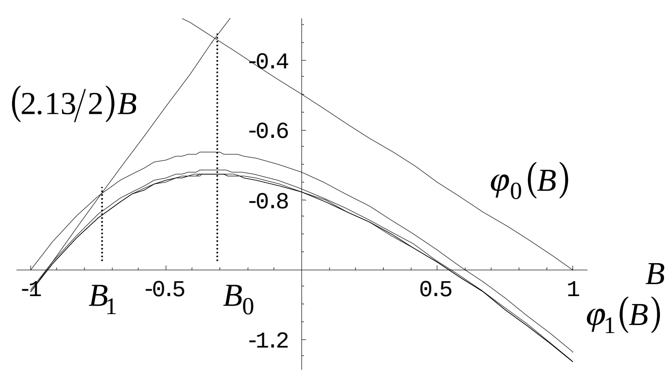

Figure 6

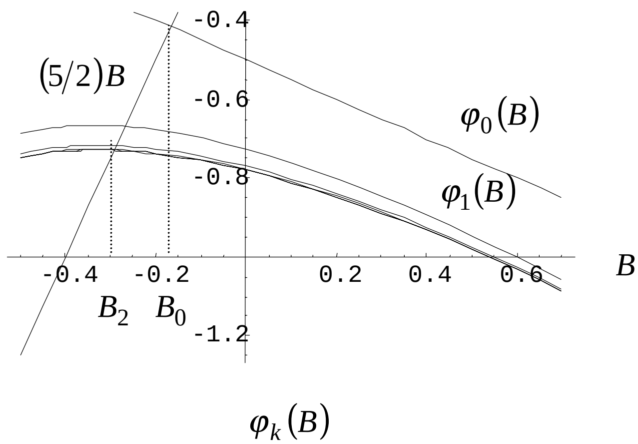
-0.4
-0.6
-0.8
-1.2
-0.4
-0.2
0.2
0.4
0.6
$(5/2)B$
$\varphi_0(B)$
$\varphi_1(B)$
B
B_2
B_0
$\varphi_k(B)$

Figure 7

Table

Critical values	$a = 2.13$	$a = 5$
B_0	-0.31949	-0.16667
B_1	-0.738161	-0.267708
B_2	-0.917673	-0.287363
B_3	-0.972434	-0.291228
B_4	-0.983001	-0.291986
B_5	-0.984362	-0.292134

Table 1